Midterm Quiz

New Beginnings Theory, Summer 2018

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Name ____

Problem 1 True/False. If true, explain why. If false, give a counterexample.

a. A valid argument can have a false conclusion. **True**
–Validity is a property of the relationship between the possible truth values of the premises to the conclusion, not the actual truth values. So it is possible to have a valid argument, one where the truth of the conclusion is conditional upon truth of the premises, with false premises and a false conclusion.

b. An invalid argument cannot have a false conclusion. **False**
–A counterexample: PSU is located in China or PSU is located in France. Therefore, PSU is located in France. From a disjunction, $P \vee Q$, we cannot validly conclude one half or the other. Note that an invalid argument *can* have a true conclusion too.

c. A pair of equivalent sentences can be derived from each other using the proof rules. **True**
–If $P \vdash Q$ and $Q \vdash P$, then we know that whenever P is true, Q is true and *vice versa*. So P and Q must be true in the same circumstances. Since P and Q must have a truth value and cannot be anything other than true or false, P and Q must be false in the same circumstances too. Whenever two sentences have the same truth value in any possible circumstances, they are equivalent.

d. The set $A = \{\emptyset\}$ contains one proper subset. **True**
–Tricky question. Every set contains the empty set as a proper subset. A is a set containing the empty set. A has two subsets: the set containing the empty set (namely, A itself) and the empty set. But only the empty set is a propert subset of A.

e. The *part of*, e.g. Oregon is a part of the USA, relation is antisymmetric and transitive. **True**
– True. The part of relation is a generally thought of as a partial order.

f. Lexicographical order, i.e. dictionary order, is a partial order but not a strict order. **True**
–Dictionary order is reflexive.

g. *Being the same age as* is an equivalence relation. **True**
–*being the same age as* is reflexive, symmetrical, and transitive.

Problem 2 Determine whether each of the statements is true. Then demonstrate either with a proof or an example (depending on what you're demonstrating).

a. $A \vee C$, $C \to D$, $D \to A \vdash A$
True. Proof:

1.	$A \vee C$	Premise
2.	$C \rightarrow D$	Premise
3.	$D \rightarrow A$	Premise
4.	$\neg A$	for IP
5.	C	1, 4 DS
6.	$\neg D$	3, 4 MT
7.	$\neg C$	2, 6 MT
8.	$\bot$	5, 7 Contradiction/$\neg$E
9.	A	4-8 IP

b. $(\neg A \wedge B) \rightarrow C$, $(A \wedge B) \rightarrow D$, $(B \wedge D) \rightarrow C$, $\vdash B \rightarrow C$
True. Proof:

1.	$(\neg A \wedge B) \rightarrow C$	Premise
2.	$(A \wedge B) \rightarrow D$	Premise
3.	$(B \wedge D) \rightarrow C$	Premise
4.	B	for $\rightarrow$I
5.	$\neg C$	for IP
6.	$\neg(B \wedge D)$	3, 5 MT
7.	$\neg B \vee \neg D$	6 DeM
8.	$\neg D$	4, 7 DS
9.	$\neg(A \wedge B)$	2, 8 MT
10.	$\neg A \vee \neg B$	9 DeM
11.	$\neg A$	4, 10 DS
12.	$\neg A \wedge B$	4, 11 $\wedge$I
13.	C	1, 12 $\rightarrow$E
14.	$\bot$	5, 13 Contradiction/$\neg$E
15.	C	5-14 IP
16.	$B \rightarrow C$	4-15 $\rightarrow$I

c. $\neg(P \rightarrow Q)$ is equivalent to $P \wedge \neg Q$
True. Proof, using only boolean identity laws:

1.	$\neg(P \rightarrow Q)$	Premise
2.	$\neg(\neg P \vee Q)$	1, MI
3.	$\neg\neg P \wedge \neg Q$	2, DeM
4.	$P \wedge \neg Q$	3, DN

d. The following set of sentences is jointly possible (logically consistent):
$A \vee B$
$A \rightarrow C$
$(A \wedge C) \rightarrow B$
$B \rightarrow (A \wedge \neg C)$
False. Proof:

1.	$A \vee B$	Premise
2.	$A \rightarrow C$	Premise
3.	$(A \wedge C) \rightarrow B$	Premise
4.	$B \rightarrow (A \wedge \neg C)$	Premise
5.	A	for IP
6.	C	2, 5 $\rightarrow$E
7.	$A \wedge C$	5, 6 $\wedge$I
8.	B	3, 7 $\rightarrow$E
9.	$A \wedge \neg C$	4, 8 $\rightarrow$E
10.	$\neg C$	9 $\wedge$E
11.	$\perp$	6, 10 contradiction/$\neg$E
12.	$\neg A$	5-11 IP
13.	B	1, 13 DS
14.	$\neg A \vee \neg\neg C$	12 $\vee$I
15.	$\neg(A \wedge \neg C)$	14 DeM
16.	$\neg B$	4, 15 MT
17.	$\perp$	13, 16 contradiction/$\neg$E

e. $(P \rightarrow Q) \vee (Q \rightarrow R)$ is necessarily true.
True. Proof:

1.	$\neg(P \rightarrow Q)$	for $\rightarrow$I
2.	$\neg(\neg P \vee Q)$	1, MI
3.	$P \wedge \neg Q$	2 DeM
4.	$\neg Q$	3 $\wedge$E
5.	$\neg Q \vee R$	4 $\vee$I
6.	$Q \rightarrow R$	5 MI
7.	$\neg(P \rightarrow Q) \rightarrow (Q \rightarrow R)$	1-6 $\rightarrow$I
8.	$(P \rightarrow Q) \vee (Q \rightarrow R)$	7 MI

f. $(P \vee Q) \rightarrow (P \wedge Q)$ is necessarily true.
False. Example: When P is true and Q is false, the sentence is false.

Problem 3 Symbolize the following arguments and determine whether valid or invalid.

a. If the earth were spherical, it would cast curved shadows on the moon. It casts curved shadows on the moon. Therefore, the earth must be spherical.
$S \rightarrow C$
C
$\therefore S$
Invalid

b. If the plasmodium parasite is found in all victims of malaria, but not in other people, then it is the source of the disease. If the plasmodium parasite is found in the anopheles mosquito and it is the source of the disease, then we should eradicate the anopheles. So we should eradicate the anopheles mosquito, if the plasmodium parasite is found in them and all victims of malaria but not in people who do not have malaria.
–Slightly different versions on the exam, which included a typo. For this version, which is valid:
$(V \wedge \neg N) \rightarrow D$
$(A \wedge D) \rightarrow E$
$\therefore (A \wedge V \wedge \neg N) \rightarrow E$

Problem 4 Suppose A, B, and C represent three bus routes through Portland. Let A, B, and C also be sets whose elements are the bus stops for the corresponding bus route. Suppose A has 25 stops, B has 30 stops, and C has 40 stops. Suppose further that A and B have 6 stops in common, A and C have 5 stops in common, B and C have 4 stops in common, and A, B, and C have 2 stops in common.

a How many distinct stops are there on the three bus routes? **82**

b How many stops for A are not stops for both B and C **23**

Problem 5 Decide which of the following functions are: injective (one-to-one), surjective (onto), both (bijective), or neither.

a $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ where $f(m, n) = m - n$ **surjective**

b $g : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ where $g(m, n) = \langle n, m\rangle$ **bijective**

c $h : \mathbb{Z} \times (\mathbb{Z} \setminus 0) \rightarrow \mathbb{Q}$ where $h(m, n) = m/n$ **surjective**

d $k : \mathbb{Z} \rightarrow \mathbb{Z}^2$ where $k(n) = \langle n, n\rangle$ **injective**

Problem 6 For each $n \in \mathbb{N}$, let $D_n = \{n, 2n, 3n, ...\}$ = multiples of n

a Find: $D_2 \cap D_7 = D_{14}$

b Find: $D_6 \cap D_8 = D_{24}$

c Find: $D_3 \cup D_{12} = D_3$

Extra Credit Prove the following identity statement is true: $(A \cup B) \times C = (A \times C) \cup (B \times C)$
–*Proof:* For the left to right direction, suppose $\langle x, y\rangle \in (A \cup B) \times C$. Then $x \in A \wedge y \in C$ or $x \in B \wedge y \in C$. In other words, $\langle x, y\rangle \in (A \times C) \cup (B \times C)$.

For the right to left direction, suppose $\langle x, y\rangle \in (A \times C) \cup (B \times C)$. Then $x \in A$ or $x \in B$, and $y \in C$. In other words, $\langle x, y\rangle \in (A \cup B) \times C$.

Extra Credit Show that if A is uncountable and B is a countable subset of A, then the set $A \setminus B$ is uncountable. (Hints: Proof by contradiction is a good idea. Also think about how set operations preserve countability.)
–*Proof:* Suppose That A is uncountable and B is countable and suppose for contradiction that $A \setminus B$ is countable. Since B is countable, $(A \setminus B) \cup B$ is countable. But then $A \subseteq (A \setminus B) \cup B)$. In other words, A is contained in a countable set, so itself must be countable. That's a contradiction, so $A \setminus B$ cannot be countable.